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The UV and IR origin of non-Abelian chiral gauge anomalies on noncommutative Minkowski spacetime

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Abstract

We discuss both the UV and IR origins of the one-loop triangle gauge anomalies for noncommutative non-Abelian chiral gauge theories with fundamental, adjoint and bi-fundamental fermions for $U(N)$ groups. We find that gauge anomalies only originate from planar triangle diagrams, the nonplanar triangle contributions giving rise to no breaking of the Ward identities. Generally speaking, theories with fundamental and bi-fundamental chiral matter are anomalous. Theories with only adjoint chiral fermions are anomaly free.

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1. Introduction

Let spacetime be noncommutative [1] Minkowski and let ψ denote a fermion chirally coupled to a $U(N)$ gauge field A_μ. Let A_μ be an $N \times N$ matrix which transforms under an infinitesimal gauge transformation as follows:

$$\left(\delta_\omega A_\mu\right)^i{}_j = \partial_\mu \omega^i{}_j - \mathrm{i}\, A^i_{\mu\, k} \star \omega^k{}_j + \mathrm{i}\, \omega^i{}_k \star A^k_{\mu\, j} \tag{1}$$

where $\omega^i{}_j = \omega^{*\,j}{}_i$, $i, j = 1, \ldots, N$, are the infinitesimal gauge transformation parameters and the symbol $\star$ represents the Moyal product of functions on Minkowski spacetime. The Moyal product is defined thus:

$$(f \star g)(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^u_\mu \partial^w_\nu} f(u) g(w)|_{u=x, w=x}$$

where $\theta^{\mu\nu}$ is an antisymmetric real matrix of either magnetic type or light-like type [2].

Following [3], we introduce three basic right-handed chiral gauge transformation laws for the fermion field

$$(\delta_\omega \psi)^i = \mathrm{i}\, \omega^i{}_j \star P_+ \psi^j \qquad \text{and} \qquad (\delta_\omega \bar\psi)_i = -\mathrm{i}\bar\psi_k \star \omega^k_i P_- \tag{2}$$

$$(\delta_\omega \psi)_j = -\mathrm{i}\, P_+ \psi_i \star \omega^i{}_j \qquad \text{and} \qquad (\delta_\omega \bar\psi)^k = \mathrm{i}\omega^k_i \star \bar\psi^i\, P_- \tag{3}$$

and

$$(\delta_\omega\psi)^i{}_j = \mathrm{i}\left(\omega^i{}_k \star P_+\,\psi^k{}_j - P_+\,\psi^i{}_k \star \omega^k{}_j\right)$$
and
$$(\delta_\omega\bar\psi)^k_i = -\mathrm{i}\left(\bar\psi^k{}_j \star \omega^j{}_i\, P_- - \omega^k{}_j \star \bar\psi^j{}_i\, P_-\right). \tag{4}$$

As usual, $P_+ = \frac{1}{2}(1+\gamma_5)$. The fermions transforming under gauge transformations as in equations (2)–(4) will be called (right-handed) fundamental, (right-handed) anti-fundamental and (right-handed) adjoint fermions, respectively.

The $U(N)$ chiral gauge theories with the fermion ψ transforming as in equations (2)–(4) are governed, respectively, by the following classical actions:

$$S = \int \mathrm{d}^4x\ \bar\psi_i \star (\mathrm{i}\partial\!\!\!/\psi^i + A^i_{\mu\,j} \star \gamma^\mu P_+\psi^j) \tag{5}$$

$$S = \int \mathrm{d}^4x\ \bar\psi^i \star (\mathrm{i}\partial\!\!\!/\psi_i - \gamma_\mu P_+\psi_j \star A^j_{\mu\,i}) \tag{6}$$

and

$$S = \int \mathrm{d}^4x\ \bar\psi^k_i \star (\mathrm{i}\partial\!\!\!/\psi^i{}_k + A^i_{\mu\,j} \star \gamma^\mu P_+\psi^j{}_k - \gamma^\mu P_+\psi^i{}_j \star A^j_{\mu\,k}). \tag{7}$$

Each action is invariant under the corresponding chiral gauge transformations; these transformations are displayed in equations (1)–(4).

The effective action, $\Gamma[A]$, which arises upon integrating out the fermionic degrees of freedom is formally given by

$$\mathrm{e}^{\mathrm{i}\Gamma[A]} = \int \mathrm{d}\psi\,\mathrm{d}\bar\psi\ \mathrm{e}^{\mathrm{i}S[A,\psi,\bar\psi]} \tag{8}$$

with $S[A,\psi,\bar\psi]$ given by any of the classical actions in equations (5)–(7). The path integral above is formally invariant under the corresponding chiral gauge transformations—see equations (1)–(4), which leads, formally, to the gauge invariance of $\Gamma[A]$. Yet, it has been shown in [4] that once the path integral is properly defined *á la* Berezin the effective action is no longer gauge invariant for fermions transforming as in equations (2) and (3), but rather the following anomaly equation holds:

$$\delta_\theta\Gamma[A] = \pm\frac{1}{24\pi^2}\,\mathrm{Tr}\int \mathrm{d}^4x\,\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\,\theta\,\partial_{\mu_1}\left[A_{\mu_2}\star\partial_{\mu_3}A_{\mu_4} - \frac{\mathrm{i}}{2}A_{\mu_2}\star A_{\mu_3}\star A_{\mu_4}\right] \tag{9}$$

where the overall + and − signs are for right-handed fundamental and right-handed anti-fundamental fermions, respectively. This equation can also be obtained by using standard diagrammatic techniques. One begins by working out the anomaly equation for the three-point contribution—the famous triangle diagrams—to $\Gamma[A]$ (the latter has been defined in equation (8)), and then one uses the Wess–Zumino consistency condition [5] to obtain the complete equation. Agreement with equation (9) demands that this triangle anomaly reads

$$\begin{aligned} p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\mathrm{eps}} &= \mp\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta \\ &\quad\times(\mathrm{Tr}\,\{T^{a_1},T^{a_2}\}\,T^{a_3}\cos\tfrac{1}{2}\theta(p_1,p_2) - \mathrm{i}\,\mathrm{Tr}\,[T^{a_1},T^{a_2}]T^{a_3}\sin\tfrac{1}{2}\theta(p_1,p_2)) \end{aligned} \tag{10}$$

where $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ gives the Fourier transform,

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = (2\pi)^4\delta(p_1+p_2+p_3)\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)$$

of the three-point function

$$\left.\frac{\delta^3\mathrm{i}\Gamma[A]}{\delta A^{a_1}_{\mu_1}(x_1)\delta A^{a_2}_{\mu_2}(x_2)\delta A^{a_3}_{\mu_3}(x_3)}\right|_{A=0} = \int\prod_{i=1}^{3}\frac{\mathrm{d}^4p_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p_ix}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) \tag{11}$$

and the superscript eps stands for the contribution to this Green function which carries the Levi-Civita pseudotensor. The indices a_1, a_2 and a_3 run over the generators of the gauge group. The symbol $\theta(p_1, p_2)$ is a shorthand for $p_{1\,\mu}\,\theta^{\mu\nu}\,p_{2\,\nu}$. Equation (10) leads clearly to the conclusion that the triangle contribution on noncommutative Minkowski for (anti-) fundamental chiral fermions is anomaly free if, and only if,

$$\mathrm{Tr}\ T^{a_1}T^{a_2}T^{a_3} = 0$$

its ordinary counterpart being $\mathrm{Tr}\,\{T^{a_1}, T^{a_2}\}T^{a_3} = 0$.

We can also have chiral gauge theories with bi-fundamental chiral fermions $\psi^{i}_{R\,j} = P_{+}\psi^{i}{}_{j}$, $i = 1, \ldots, N$ and $j = 1, \ldots, M$ [3]. Now the fermion couples to a $U(N)$ gauge field, say A_μ, and a $U(M)$ gauge field, say B_μ, the former being an $N \times N$ matrix and the latter an $M \times M$ matrix. The classical action for this theory reads

$$S = \int \mathrm{d}^4x\ \bar{\psi}^{k}_{i} \star (\mathrm{i}\partial\!\!\!/\psi^{i}{}_{k} + A^{i}_{\mu\,j} \star \gamma^{\mu} P_{+}\psi^{j}{}_{k} - \gamma^{\mu} P_{+}\psi^{i}{}_{j} \star B^{j}_{\mu\,k}). \tag{12}$$

This action is invariant under the following infinitesimal gauge transformations:

$$\begin{aligned}
(\delta_{(\omega,\chi)}\psi)^{i}{}_{j} &= \mathrm{i}(\omega^{i}{}_{j} \star P_{+}\psi^{j}{}_{i} - P_{+}\psi^{i}{}_{j} \star \chi^{j}_{i})\\
(\delta_{(\omega,\chi)}\bar{\psi})^{k}_{i} &= -\mathrm{i}(\bar{\psi}^{k}{}_{j} \star \omega^{j}_{i} P_{-} - \chi^{k}_{j} \star \bar{\psi}^{j}_{i} P_{-})\\
(\delta_{\omega}A_{\mu})^{i}{}_{j} &= \partial_{\mu}\omega^{i}{}_{j} - \mathrm{i}\,A^{i}_{\mu\,k} \star \omega^{k}{}_{j} + \mathrm{i}\,\omega^{i}{}_{k} \star A^{k}_{\mu\,j}\\
(\delta_{\chi}B_{\mu})^{i}{}_{j} &= \partial_{\mu}\chi^{i}{}_{j} - \mathrm{i}\,B^{i}_{\mu\,k} \star \chi^{k}_{j} + \mathrm{i}\,\chi^{i}_{k} \star B^{k}_{\mu\,j}
\end{aligned}$$

where $\omega^{i}{}_{j} = \omega^{*\,j}{}_{i}$, $i, j = 1, \ldots, N$, and $\chi^{i}{}_{j} = \chi^{*\,j}{}_{i}$, $i, j = 1, \ldots, M$, are the infinitesimal gauge transformation parameters.

The effective action, $\Gamma[A, B]$, that one obtains by integrating over the fermionic degrees of freedom formally reads thus:

$$\mathrm{e}^{\mathrm{i}\Gamma[A,B]} = \int \mathrm{d}\psi\,\mathrm{d}\bar{\psi}\ \mathrm{e}^{\mathrm{i}S[A,B,\psi,\bar{\psi}]} \tag{13}$$

with $S[A, B, \psi, \bar{\psi}]$ given in equation (12). We shall see that in general there are triangle gauge anomalies jeopardizing the formal gauge invariance of $\Gamma[A, B]$.

It is well known that the chiral gauge anomaly on ordinary Minkowski spacetime can be understood either as a short-distance phenomenon (UV effect) [6] or as an IR effect (large-distance phenomenon) [7]. The purpose of this paper is to show that non-Abelian chiral anomalies on noncommutative Minkowski spacetime can also be explained as either a UV effect or an IR phenomenon. Recall that if the chiral fermions of the theory are either adjoint or bi-fundamental, there are non-planar contributions to the three-point function of the effective action ($\Gamma_{\mathrm{adj}}[A]$ and $\Gamma[A, B]$ in equations (8) and (13)) and one wonders whether these non-planar contributions may give rise to some gauge anomaly due to its noncommutative IR structure; this structure being a consequence of their being regularized in the UV by the appropriate Moyal exponentials [8]. We shall show in this paper that, at least for the theories we have studied, there are no anomalous contributions from the nonplanar triangle diagrams: gauge anomalies—if they exist—are due to planar triangle diagrams. We have assumed that, as in the ordinary case, true anomalies always involve the Levi-Civita pseudotensor. Standard arguments [9] can be put forward to support this assumption.

The layout of this paper is as follows. Section 2 is devoted to the analysis of the anomaly equation—equation (10)—as a UV effect. In this section we shall also show that the chiral gauge theory whose classical action is given in equation (7) is anomaly free. We close the section by computing the triangle gauge anomalies for a chiral theory with a bi-fundamental right-handed fermion and conclude that they only originate from the planar contribution to its

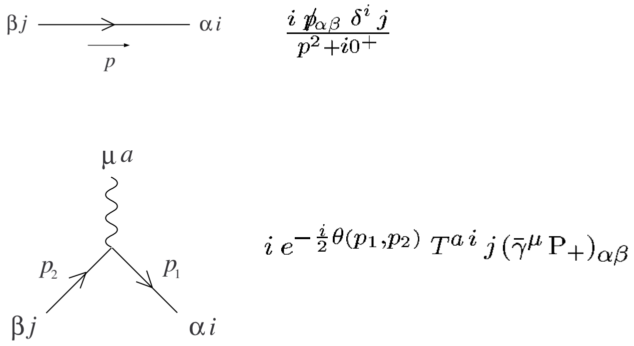

Figure 1. Feynmann rules.

effective action; the nonplanar part being thus anomaly free. In section 3 we shall exhibit the IR origin of the non-Abelian chiral anomalies we have worked out in section 2. We include an appendix with the relevant Feynman integrals.

2. The UV origin of non-Abelian chiral gauge anomalies

Let us begin with the chiral theory whose action is given by equation (5). The UV character of equation (10) is made apparent by computing its lhs with the help of a regularization method. We shall use dimensional regularization as defined by Breitenlohner and Maison [10] (see [11] for an alternative), i.e. with the definition of γ_5 given by 't Hooft and Veltman, and take the following classical action in the 'd-dimensional' space of dimensional regularization (see [12] and references therein):

$$S = \int \mathrm{d}^d x\ \bar{\psi}_i \star (\mathrm{i}\not{\partial}\psi^i + A^a_\mu T^{a\,i}{}_j \bar{\gamma}^\mu P_+ \star \psi^j).$$

Here, $T^{a\,i}{}_j = T^{*\,a\,j}{}_i$. The object denoted by the symbol $\bar{\gamma}^\mu$ and the other objects in the algebra of 'd-dimensional' covariants are defined as in section 2 of [12]. The 'd-dimensional' counterpart of $\theta^{\mu\nu}$ is defined as an object which satisfies

$$\theta^{\mu\nu} = -\theta^{\nu\mu} \qquad \hat{g}_{\mu\rho}\theta^{\rho\nu} = 0 \qquad p_\mu\, \theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\, p_\nu \geqslant 0 \qquad \forall p_\mu.$$

The Feynman rules needed to reproduce our computations are given in figure 1.

Let us define the dimensionally regularized counterpart of the lhs of equation (10):

$$\Delta^{a_1a_2a_3}_{\mu_1\mu_2}(p_1, p_2; d) = p_3^{\mu_3}\, \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2; d)^{\mathrm{eps}}.$$

At the one-loop level $\Delta^{a_1a_2a_3}_{\mu_1\mu_2}(p_1, p_2; d)$ is given by the sum of the contributions from the two triangle diagrams in figure 2. This sum reads

$$\begin{aligned}\Delta^{a_1a_2a_3}_{\mu_1\mu_2}(p_1, p_2; d) = {}& \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\ \mathrm{Tr}\ T^{a_1}T^{a_2}T^{a_3}\ \Delta^{(1)}_{\mu_1\mu_2}(p_1, p_2; d)\\ &+\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\ \mathrm{Tr}\ T^{a_2}T^{a_1}T^{a_3}\ \Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d)\end{aligned} \tag{14}$$

with

$$\Delta^{(1)}_{\mu_1\mu_2}(p_1, p_2; d) = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}^{\mathrm{eps}}\left\{(\not{q} + \not{p}_1)\, \bar{\gamma}_{\mu_1} P_+ \not{q} \bar{\gamma}_{\mu_2} P_+ (\not{q} - \not{p}_2)(\bar{\not{p}}_1 + \bar{\not{p}}_2) P_+\right\}}{(q^2 + \mathrm{i}0^+)((q + p_1)^2 + \mathrm{i}0^+)((q - p_2)^2 + \mathrm{i}0^+)}$$

and

$$\Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d) = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}^{\mathrm{eps}}\left\{(\not{q} + \not{p}_2)\, \bar{\gamma}_{\mu_2} P_+ \not{q} \bar{\gamma}_{\mu_1} P_+ (\not{q} - \not{p}_1)(\bar{\not{p}}_1 + \bar{\not{p}}_2) P_+\right\}}{(q^2 + \mathrm{i}0^+)((q + p_2)^2 + \mathrm{i}0^+)((q - p_1)^2 + \mathrm{i}0^+)}. \tag{15}$$

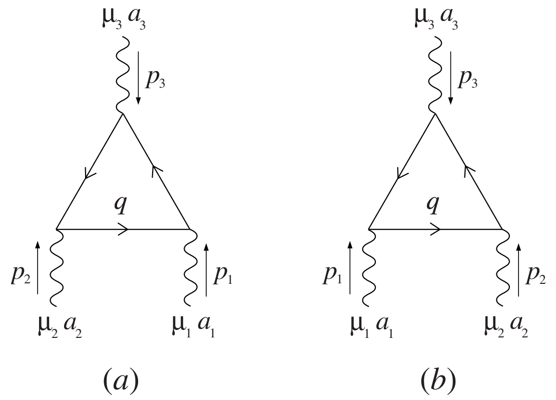

Figure 2. Triangle diagrams.

In the previous equation $\mathrm{tr}^{\,\mathrm{eps}}$ shows that only contributions involving the Levi-Civita symbol $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ are kept upon computing the trace over the gammas.

Now, the Feynman diagrams in figure 2 are planar; hence, it can be readily seen [13] that their noncommutative character is completely embodied (see equation (14)) in the overall phase factors $\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}$ and $\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}$. Then, it does not come as a surprise that equation (10) holds, for the Feynman integrals in equation (15) are the standard integrals whose UV behaviour gives rise to the non-Abelian chiral anomaly on commutative Minkowski space.

Taking into account that $P_+\hat{\gamma}_\mu\bar{\gamma}_\nu P_+ = 0$ and performing some standard manipulations one shows that

$$\begin{aligned}\Delta^{(1)}_{\mu_1\mu_2}&(p_1,p_2;d)\\ &= -\frac{1}{2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{tr}\left\{\bar{\gamma}_\alpha\bar{\gamma}_{\mu_1}\bar{\gamma}_\beta\bar{\gamma}_{\mu_2}\gamma_5\right\}\bar{p}_1^\alpha(\bar{q}+\bar{p}_2)^\beta\,\hat{q}^2}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)((q+p_1+p_2)^2+\mathrm{i}0^+)}\\ &\quad -\frac{1}{2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{tr}\left\{\bar{\gamma}_{\mu_1}\bar{\gamma}_\alpha\bar{\gamma}_{\mu_2}\bar{\gamma}_\beta\gamma_5\right\}(\bar{q}+\bar{p}_1)^\alpha\,\bar{p}_2^\beta\,\hat{q}^2}{(q^2+\mathrm{i}0^+)((q+p_1)^2+\mathrm{i}0^+)((q+p_1+p_2)^2+\mathrm{i}0^+)}.\end{aligned}\qquad (16)$$

Notice that the integrand of the integrals in equation (16) formally vanishes in the limit $d\to 4$, since it contains the evanescent term $\hat{q}^2$. However, the limit $d\to 4$ of these integrals although finite is not zero. Indeed, if we take into account that $\hat{q}^2 = q^\alpha q^\beta \hat{g}_{\alpha\beta}$, we readily see that what we are facing is the computation of integrals which are UV divergent by power-counting at $d=4$ and which will develop a simple pole at $d=4$ when computed in dimensional regularization (notice that the integrals at hand are IR finite by power-counting at nonexceptional momenta). This pole will be cancelled at the end of the day by the evanescent (order $d-4$) contribution from the contraction with $\hat{g}_{\alpha\beta}$, yielding a polynomial in the external momenta (short-distance operator) as the value for $\Delta^{(1)}_{\mu_1\mu_2}(p_1,p_2;d)$ at $d=4$. We have thus explained the non-Abelian chiral anomaly of equation (10) as a UV effect. Indeed, a little computation shows that the integrals in equation (16) yield the following result:

$$\Delta^{(1)}_{\mu_1\mu_2}(p_1,p_2;d=4) = -\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta. \qquad (17)$$

The reader may find the following integrals useful:

$$\int \frac{d^d q}{(2\pi)^d} \frac{\hat{q}^2}{q^2\,(q+p_2)^2\,(q+p_1+p_2)^2} = -\frac{i}{16\pi^2}\left(\frac{1}{2}\right) + O(d-4)$$
$$\int \frac{d^d q}{(2\pi)^d} \frac{\hat{q}^2\,\bar{q}^\alpha}{q^2\,(q+p_2)^2\,(q+p_1+p_2)^2} = \frac{i}{16\pi^2}\left(\frac{1}{6}\right)(\bar{p}_1+2\bar{p}_2)^\alpha + O(d-4).$$

It is clear that for $\Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d)$ in equation (15) one will obtain the following finite answer:

$$\Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d=4) = -\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha\,p_2^\beta. \tag{18}$$

Finally, if we substitute this result and equation (17) in (14), we shall recover the one-loop triangle anomaly of equation (10).

A completely similar analysis can be done for the chiral theory defined by the action in equation (6). Let us move on and compute $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2)^{\rm eps}$ for the theory with adjoint fermionic matter. The classical action of this theory is given in equation (7). The Ward identity that should hold if the gauge symmetry of the classical theory is a symmetry of the quantum theory reads

$$\int d^4x\,\omega^{i_1}_{i_2}\star\partial_\mu\frac{\delta\Gamma[A]}{\delta A^{i_1}_{\mu\,i_2}} = i\int d^4x\,\omega^{i_1}_{i_2}\star\left[A^{i_2}_{\mu\,i_3}\star\frac{\delta\Gamma[A]}{\delta A^{i_1}_{\mu\,i_3}} - \frac{\delta\Gamma[A]}{\delta A^{i_3}_{\mu\,i_2}}\star A^{i_3}_{\mu\,i_1}\right]. \tag{19}$$

It can be readily shown that the previous equation holds by writing the classical action of the theory in terms of Majorana fermions with gamma matrices in the Majorana representation. Indeed, in so doing the coupling to the gauge field is vector-like so no gauge anomaly occurs, and yet we shall carry out explicit computations for the classical action written as in equation (7), since the intermediate results we shall obtain will be useful when we consider bi-fundamental chiral matter coupled to gauge fields.

For $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2)^{\rm eps}$, equation (19) boils down to

$$p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2)^{\rm eps} = 0.$$

To obtain this equation it is necessary to take into account that the two-point contribution to $\Gamma[A]$ has no pseudotensor contribution.

The dimensional regularization counterpart of the action in equation (7) will have for us the following expression:

$$S = \int d^d x\;\bar{\psi}^k_{\;i}\star(i\partial\!\!\!/\psi^i_{\;k} + A^i_{\mu\,j}\star\bar{\gamma}^\mu P_+\psi^j_{\;k} - \bar{\gamma}^\mu P_+\psi^i_{\;j}\star A^j_{\mu\,k})$$

with the same notation as at the beginning of this section. Instead of deriving Feynman rules from this action and computing $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2)^{\rm eps}$ from the corresponding triangle diagrams in figure 2, we shall follow an alternative procedure, which will supply a more thorough understanding of the final answer. Let us introduce first the following chiral current in the 'd-dimensional' space of dimensional regularization:

$$j^a_\mu(x) \equiv i\frac{\delta S[A]}{A^a_\mu(x)} \equiv j^{a\,-}_{\;\mu}(x) + j^{a\,+}_{\;\mu}(x) \tag{20}$$

where

$$j^{a\,-}_{\;\mu}(x) = -i\,\psi^j_{\;k\,\beta}\star\bar{\psi}^k_{\;i\,\alpha}(x)\,T^{a\,i}_{\;\;j}\,(\bar{\gamma}^\mu P_+)_{\alpha\beta}$$

and

$$j^{a\,+}_{\;\mu}(x) = -i\,\bar{\psi}^k_{\;i\,\alpha}\star\psi^i_{\;j\,\beta}(x)T^{a\,j}_{\;\;k}\,(\bar{\gamma}^\mu P_+)_{\alpha\beta}. \tag{21}$$

Let $j^{a\,(\cdot)}_{\mu}(p)$ be given by

$$j^{a\,(\cdot)}_{\mu}(x) = \int \frac{d^4 p}{(4\pi)^4}\, e^{ipx}\, j^{a\,(\cdot)}_{\mu}(p).$$

Then, the three-point function (equation (11)) in momentum space reads

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = \langle j^{a_1}_{\ \mu_1}(p_1)\, j^{a_2}_{\ \mu_2}(p_2)\, j^{a_3}_{\ \mu_3}(p_3)\rangle_{\rm con}$$

where the subscript 'con' refers to the connected part of the corresponding Green function. Throughout this paper, vacuum expectation values are computed with the free fermionic action. Taking into account equation (20), we obtain

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)_{\rm P} + \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)_{\rm NP}$$

where

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)_{\rm P} = \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle_{\rm con} + \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle_{\rm con} \tag{22}$$

and

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}&(p_1, p_2, p_3)_{\rm NP}\\
&= \langle j^{a_1\,-}_{\ \mu_1}(p_1) j^{a_2\,-}_{\ \mu_2}(p_2) j^{a_3\,+}_{\ \mu_3}(p_3)\rangle_{\rm con} + \langle j^{a_1\,+}_{\ \mu_1}(p_1) j^{a_2\,+}_{\ \mu_2}(p_2) j^{a_3\,-}_{\ \mu_3}(p_3)\rangle_{\rm con}\\
&\quad\times\langle j^{a_1\,-}_{\ \mu_1}(p_1) j^{a_2\,+}_{\ \mu_2}(p_2) j^{a_3\,-}_{\ \mu_3}(p_3)\rangle_{\rm con} + \langle j^{a_1\,+}_{\ \mu_1}(p_1) j^{a_2\,-}_{\ \mu_2}(p_2) j^{a_3\,+}_{\ \mu_3}(p_3)\rangle_{\rm con}\\
&\quad\times\langle j^{a_1\,+}_{\ \mu_1}(p_1) j^{a_2\,-}_{\ \mu_2}(p_2) j^{a_3\,-}_{\ \mu_3}(p_3)\rangle_{\rm con} + \langle j^{a_1\,-}_{\ \mu_1}(p_1) j^{a_2\,+}_{\ \mu_2}(p_2) j^{a_3\,+}_{\ \mu_3}(p_3)\rangle_{\rm con}.\end{aligned} \tag{23}$$

The subscripts 'P' and 'NP' refer, respectively, to the planar and nonplanar parts of $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)$. The reader may easily realize that only when the three currents in the correlation function carry the same superscript, $-$ or $+$, there is no Moyal exponential carrying the loop momenta. Notice that each correlation function of the type $\langle j\ j\ j\rangle^{\rm con}$ above can be interpreted as the sum of two triangle diagrams with vertices given by the currents of the former.

Now, taking into account equation (21), it can be easily shown that the the Green functions contributing to the rhs of equation (22) satisfy

$$\begin{aligned}p_3^{\mu_3}\, \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\rm eps}_{\rm con} &= (2\pi)^4\delta(p_1+p_2+p_3)\, N\\
&\quad\times e^{-\frac{i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_1}T^{a_2}T^{a_3}\ \Delta^{(1)}_{\mu_1\mu_2}(p_1, p_2; d)\\
&\quad+e^{\frac{i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_2}T^{a_1}T^{a_3}\ \Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d)\\
p_3^{\mu_3}\, \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\rm eps}_{\rm con} &= -(2\pi)^4\delta(p_1+p_2+p_3)\, N\\
&\quad\times e^{-\frac{i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_1}T^{a_2}T^{a_3}\ \Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d)\\
&\quad+e^{\frac{i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_2}T^{a_1}T^{a_3}\ \Delta^{(1)}_{\mu_1\mu_2}(p_1, p_2; d)\end{aligned} \tag{24}$$

where $\Delta^{(1)}_{\mu_1\mu_2}(p_1, p_2; d)$ and $\Delta^{(2)}_{\mu_1\mu_2}(p_1, p_2; d)$ are given in equation (15) and the superscript 'eps' indicates that one should keep only contributions involving the Levi-Civita symbol. Now, substituting equations (17) and (18) into (24), one obtains that the following equations hold at $d = 4$:

$$\begin{aligned}p_3^{\mu_3}\, \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\rm eps}_{\rm con} &= -(2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta\\
&\quad\times N\big({\rm Tr}\,\{T^{a_1}, T^{a_2}\}\, T^{a_3}\cos\tfrac{1}{2}\theta(p_1, p_2) - i\,{\rm Tr}\,[T^{a_1}, T^{a_2}]T^{a_3}\sin\tfrac{1}{2}\theta(p_1, p_2)\big)\\
p_3^{\mu_3}\, \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\rm eps}_{\rm con} &= (2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta\\
&\quad\times N\big({\rm Tr}\,\{T^{a_1}, T^{a_2}\}\, T^{a_3}\cos\tfrac{1}{2}\theta(p_1, p_2) + i\,{\rm Tr}\,[T^{a_1}, T^{a_2}]T^{a_3}\sin\tfrac{1}{2}\theta(p_1, p_2)\big).\end{aligned}$$

Hence, each correlation function of currents contributing to the rhs of equation (22) yields an anomalous term, but its sum, i.e. the planar part of $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)^{\rm eps}$, carries no anomaly:

$$p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)^{\rm eps}_{\rm P} = 0.$$

The reader should notice that the result we have just derived can be understood as follows: the sum of the two triangle diagrams contributing to $\langle j^{a_1\,-}_{\mu_1}(p_1)\, j^{a_2\,-}_{\mu_2}(p_2)\, j^{a_3\,-}_{\mu_3}(p_3)\rangle_{\rm con}$ yields a chiral anomaly opposite to the chiral anomaly from the sum of the two triangle diagrams contributing to $\langle j^{a_1\,+}_{\mu_1}(p_1)\, j^{a_2\,+}_{\mu_2}(p_2)\, j^{a_3\,+}_{\mu_3}(p_3)\rangle_{\rm con}$; i.e., the contribution from the fermionic modes in the fundamental representation of $U(N)$ moving around the loop cancels the contribution furnished by the fermionic modes in the anti-fundamental representation of $U(N)$ propagating along the loop: recall that the adjoint representation of $U(N)$ can be understood as the product of its fundamental and anti-fundamental representations.

Let us now show that there is no anomaly in the pseudotensor part of the nonplanar contribution given in equation (23). Here, of course, we shall meet only integrals which give UV finite results at $d = 4$—since the Moyal exponential regulates them in the UV—but which develop, as a consequence of the UV/IR connection in noncommutative field theories, IR divergences as one approaches the noncommutative IR region $\tilde p = 0$. Let us see whether or not they carry any anomaly. For the first three-current correlation function on the rhs of equation (23), one obtains the following intermediate results at $d = 4$:

$$\begin{aligned} p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\, j^{a_2\,-}_{\mu_2}(p_2)\, j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} &= (2\pi)^4\delta(p_1+p_2+p_3)\,{\rm Tr}(T^{a_1}T^{a_2})\,{\rm Tr}\,T^{a_3} \\ &\quad\times\left[{\rm e}^{\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) + {\rm e}^{-\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3)\right] \end{aligned} \tag{25}$$

with

$$\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) = \int \frac{{\rm d}^4q}{(2\pi)^4}\,{\rm e}^{-{\rm i}\theta(q,p_3)}\,\frac{{\rm tr}^{\rm eps}\left\{(\not q+\not p_1)\,\gamma_{\mu_1}P_+\not q\gamma_{\mu_2}P_+\,(\not q-\not p_2)(\not p_1+\not p_2)P_+\right\}}{(q^2+{\rm i}0^+)((q+p_1)^2+{\rm i}0^+)((q-p_2)^2+{\rm i}0^+)}$$

and

$$\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) = \int \frac{{\rm d}^4q}{(2\pi)^4}\,{\rm e}^{-{\rm i}\theta(q,p_3)}\,\frac{{\rm tr}^{\rm eps}\left\{(\not q+\not p_2)\,\gamma_{\mu_2}P_+\not q\gamma_{\mu_1}P_+\,(\not q-\not p_1)(\not p_1+\not p_2)P_+\right\}}{(q^2+{\rm i}0^+)((q+p_2)^2+{\rm i}0^+)((q-p_1)^2+{\rm i}0^+)}.$$

In the previous integrals $p_1+p_2+p_3 = 0$. Notice the characteristic Moyal factor, ${\rm e}^{-{\rm i}\theta(q,p_3)}$, of a nonplanar contribution. The integrals are well defined provided we are off the noncommutative IR region defined by $\tilde p_3^2 = 0$. Let us show now that

$${\rm e}^{\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) + {\rm e}^{-\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) = 0. \tag{26}$$

If we change variables $q\to q+p_2$ and $q\to q+p_1$ in $\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3)$ and $\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3)$, respectively, and use the cyclicity of the trace, we obtain

$$\begin{aligned} &{\rm e}^{\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) + {\rm e}^{-\frac{\rm i}{2}\theta(p_1,p_2)}\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_3) \\ &\quad = {\rm e}^{-\frac{\rm i}{2}\theta(p_1,p_2)}\int\frac{{\rm d}^4q}{(2\pi)^4}\,{\rm e}^{-{\rm i}\theta(q,p_3)} \\ &\qquad\times{\rm tr}^{\rm eps}\left\{\frac{1}{\not q}(\not p_1+\not p_2)P_+\,\frac{1}{\not q+\not p_1+\not p_2}\,\gamma_{\mu_1}P_+\,\frac{1}{\not q+\not p_2}\,\gamma_{\mu_2}P_+\right\} \\ &\qquad+{\rm e}^{\frac{\rm i}{2}\theta(p_1,p_2)}\int\frac{{\rm d}^4q}{(2\pi)^4}\,{\rm e}^{-{\rm i}\theta(q,p_3)} \\ &\qquad\times{\rm tr}^{\rm eps}\left\{\frac{1}{\not q}(\not p_1+\not p_2)P_+\,\frac{1}{\not q+\not p_1+\not p_2}\,\gamma_{\mu_2}P_+\,\frac{1}{\not q+\not p_1}\,\gamma_{\mu_1}P_+\right\}. \end{aligned} \tag{27}$$

Now, using the equations $\gamma_\mu\gamma_\nu P_+ = P_+\gamma_\mu\gamma_\nu$, $P_+^2 = P_+$ and $(\not p_1 + \not p_2)\gamma_5 = -\not q\gamma_5 - \gamma_5(\not q + \not p_1 + \not p_2)$ one readily casts the rhs of equation (27) into the form

$$\begin{aligned}
&-\frac{1}{2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}\theta(q,p_3)}\left[\mathrm{tr}\left\{\gamma_5\,\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_1}\,\frac{1}{\not q + \not p_2}\,\gamma_{\mu_2}\right\}\right.\\
&\qquad\left.+\,\mathrm{tr}\left\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_1}\,\frac{1}{\not q + \not p_2}\,\gamma_{\mu_2}\right\}\right]\\
&-\frac{1}{2}\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}\theta(q,p_3)}\left[\mathrm{tr}\left\{\gamma_5\,\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_2}\,\frac{1}{\not q + \not p_1}\,\gamma_{\mu_1}\right\}\right.\\
&\qquad\left.+\,\mathrm{tr}\left\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_2}\,\frac{1}{\not q + \not p_1}\,\gamma_{\mu_1}\right\}\right].
\end{aligned}\tag{28}$$

Some Dirac algebra leads, respectively, to the following expressions:

$$\begin{aligned}
&\mathrm{tr}\left\{\gamma_5\,\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_1}\,\frac{1}{\not q + \not p_2}\,\gamma_{\mu_2}\right\} = -\,\mathrm{tr}\left\{\frac{1}{\not q + \not p_2}\,\gamma_5\gamma_{\mu_2}\,\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_1}\right\}\\
&\mathrm{tr}\left\{\gamma_5\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_2}\,\frac{1}{\not q + \not p_1}\,\gamma_{\mu_1}\right\} = -\,\mathrm{tr}\left\{\frac{1}{\not q + \not p_1}\,\gamma_5\gamma_{\mu_1}\frac{1}{\not q + \not p_1 + \not p_2}\,\gamma_{\mu_2}\right\}.
\end{aligned}\tag{29}$$

Substituting these equations in equation (28) and performing appropriate momentum shifts, one easily shows that, in equation (28), the first integral cancels the fourth integral and the second integral cancels the third one, thus proving that the equation (26) actually holds. We obtain finally

$$p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\,j^{a_2\,-}_{\mu_2}(p_2)\,j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0 \tag{30}$$

a result which is obtained by substituting equation (26) in (25). The same conclusion can be reached, using completely analogous methods, for the three-current correlation function $\langle j^{a_1\,+}_{\mu_1}(p_1)j^{a_2\,+}_{\mu_2}(p_2)j^{a_3\,-}_{\mu_3}(p_3)\rangle_{\mathrm{con}}$:

$$p_3^{\mu_3}\,\langle j^{a_1\,+}_{\mu_1}(p_1)\,j^{a_2\,+}_{\mu_2}(p_2)\,j^{a_3\,-}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0. \tag{31}$$

Things do not work the same way for the remaining Green functions on the rhs of equation (23). Actually, each three-current correlation function gives a contribution, vanishing the sum of them all. Let us see this. Some algebra leads to

$$\begin{aligned}
p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\,j^{a_2\,+}_{\mu_2}(p_2)\,j^{a_3\,-}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= (2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}(T^{a_1}T^{a_3})\,\mathrm{Tr}\;T^{a_2}\\
&\quad\times\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2) + \mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2)\right]\\
p_3^{\mu_3}\,\langle j^{a_1\,+}_{\mu_1}(p_1)\,j^{a_2\,-}_{\mu_2}(p_2)\,j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= -(2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}(T^{a_1}T^{a_3})\,\mathrm{Tr}\;T^{a_2}\\
&\quad\times\left[\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2) + \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2)\right]\\
p_3^{\mu_3}\,\langle j^{a_1\,+}_{\mu_1}(p_1)\,j^{a_2\,-}_{\mu_2}(p_2)\,j^{a_3\,-}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= (2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}(T^{a_2}T^{a_3})\,\mathrm{Tr}\;T^{a_1}\\
&\quad\times\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1) + \mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1)\right]\\
p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\,j^{a_2\,+}_{\mu_2}(p_2)\,j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= -(2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}(T^{a_2}T^{a_3})\,\mathrm{Tr}\;T^{a_1}\\
&\quad\times\left[\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1) + \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1)\right].
\end{aligned}\tag{32}$$

In this equation, the contributions denoted by $\Delta^{(1)\pm}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i)$ and $\Delta^{(2)\pm}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i)$, with $i = 1$ and 2, are given by the following integrals:

$$\Delta^{(1)\pm}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_i) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\pm\mathrm{i}\theta(q,p_i)}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\left\{(\not{q}+\not{p}_1)\,\gamma_{\mu_1}P_+\not{q}\gamma_{\mu_2}P_+(\not{q}-\not{p}_2)(\not{p}_1+\not{p}_2)P_+\right\}}{(q^2+\mathrm{i}0^+)((q+p_1)^2+\mathrm{i}0^+)((q-p_2)^2+\mathrm{i}0^+)}$$

and

$$\Delta^{(2)\pm}_{\mu_1\mu_2}(p_1, p_2\tilde{p}_i) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\pm\mathrm{i}\theta(q,p_i)}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\left\{(\not{q}+\not{p}_2)\,\gamma_{\mu_2}P_+\not{q}\gamma_{\mu_1}P_+(\not{q}-\not{p}_1)(\not{p}_1+\not{p}_2)P_+\right\}}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)((q-p_1)^2+\mathrm{i}0^+)}.$$

Using the same variety of tricks that led to equation (26), one shows that now

$$\begin{aligned}&\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_i) + \mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_i)\\&\quad = \mp 4\sin\frac{1}{2}\theta(p_1,p_2)\,\varepsilon_{\mu_1\mu_2\alpha\beta}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_i)}\,\frac{q^\alpha p_i^\beta}{(q^2+\mathrm{i}0^+)((q+p_i)^2+\mathrm{i}0^+)}\end{aligned}\tag{33}$$

where $i = 1$ and 2. For the sake of the reader, we shall spell out the computations leading to the previous equation. Let us change variables $q \to q + p_2$ and $q \to q + p_1$ in $\Delta^{(1)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_2)$ and $\Delta^{(2)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_2)$, respectively, and use the cyclicity of the trace, to obtain

$$\begin{aligned}&\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_2) + \mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)\mp}_{\mu_1\mu_2}(p_1, p_2|\tilde{p}_2)\\&\quad = \mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\\&\qquad\times\mathrm{tr}^{\,\mathrm{eps}}\left\{\frac{1}{\not{q}}(\not{p}_1+\not{p}_2)P_+\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\gamma_{\mu_1}P_+\frac{1}{\not{q}+\not{p}_2}\gamma_{\mu_2}P_+\right\}\\&\qquad+\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\\&\qquad\times\mathrm{tr}^{\,\mathrm{eps}}\left\{\frac{1}{\not{q}}(\not{p}_1+\not{p}_2)P_+\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\gamma_{\mu_2}P_+\frac{1}{\not{q}+\not{p}_1}\gamma_{\mu_1}P_+\right\}.\end{aligned}\tag{34}$$

Notice that, unlike equation (27), the exponential factor in front of each integral is the same. This will turn out to be of the utmost importance. Next, let us use the equations $\gamma_\mu\gamma_\nu P_+ = P_+\gamma_\mu\gamma_\nu$, $P_+^2 = P_+$ and

$$(\not{p}_1+\not{p}_2)\gamma_5 = -\not{q}\gamma_5 - \gamma_5(\not{q}+\not{p}_1+\not{p}_2)$$

to cast the rhs of equation (34) into the form

$$\begin{aligned}&-\frac{1}{2}\,\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\Bigg[\mathrm{tr}\left\{\gamma_5\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\gamma_{\mu_1}\frac{1}{\not{q}+\not{p}_2}\gamma_{\mu_2}\right\}\\&\qquad+\mathrm{tr}\left\{\frac{1}{\not{q}}\gamma_5\gamma_{\mu_1}\frac{1}{\not{q}+\not{p}_2}\gamma_{\mu_2}\right\}\Bigg]\\&-\frac{1}{2}\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\Bigg[\mathrm{tr}\left\{\gamma_5\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\gamma_{\mu_2}\frac{1}{\not{q}+\not{p}_1}\gamma_{\mu_1}\right\}\\&\qquad+\mathrm{tr}\left\{\frac{1}{\not{q}}\gamma_5\gamma_{\mu_2}\frac{1}{\not{q}+\not{p}_1}\gamma_{\mu_1}\right\}\Bigg].\end{aligned}\tag{35}$$

Recall that $\mathrm{tr}^{\,\mathrm{eps}}$ means that one only keeps contributions that carry the Levi-Civita symbol. Taking into account equation (29), one obtains that equation (35) can be written as follows:

$$\begin{aligned}
&-\frac{1}{2}\,\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\left[-\,\mathrm{tr}\left\{\frac{1}{\not q+\not p_2}\gamma_5\,\gamma_{\mu_2}\,\frac{1}{\not q+\not p_1+\not p_2}\gamma_{\mu_1}\right\}\right.\\
&\qquad\left.+\,\mathrm{tr}\left\{\frac{1}{\not q}\gamma_5\gamma_{\mu_1}\,\frac{1}{\not q+\not p_2}\gamma_{\mu_2}\right\}\right]\\
&-\frac{1}{2}\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\left[-\,\mathrm{tr}\left\{\frac{1}{\not q+\not p_1}\gamma_5\,\gamma_{\mu_1}\,\frac{1}{\not q+\not p_1+\not p_2}\gamma_{\mu_2}\right\}\right.\\
&\qquad\left.+\,\mathrm{tr}\left\{\frac{1}{\not q}\gamma_5\gamma_{\mu_2}\,\frac{1}{\not q+\not p_1}\gamma_{\mu_1}\right\}\right].
\end{aligned}\tag{36}$$

Let us next make the shifts $q\to q-p_2$ and $q\to q-p_1$ in the first and third integrals in equation (36). Then, we readily see that the first integral cancels the fourth integral of equation (36), but the sum of the second and third integrals of equation (36) yields

$$\frac{1}{2}\left(\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta(p_1,p_2)}-\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\right)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{e}^{\mp\mathrm{i}\theta(q,p_2)}\;\mathrm{tr}\left\{\frac{1}{\not q}\gamma_5\gamma_{\mu_1}\,\frac{1}{\not q+\not p_2}\gamma_{\mu_2}\right\}.$$

From this equation one obtains equation (33) for $i=2$. Let us now replace the integral in equation (33) with its value, which can be found in the appendix. One obtains, for $i=2$, that

$$\begin{aligned}
&\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(1)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i)+\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\Delta^{(2)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i)=\frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\\
&\qquad\times\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde p_i^{\alpha}\,p_i^{\beta}}{\tilde p_i^2}\int_0^1\mathrm{d}x\,\sqrt{\tilde p_i^2(-p_i^2-\mathrm{i}0^+)x(1-x)}\\
&\qquad\times K_1\left(\sqrt{\tilde p_i^2(-p_i^2-\mathrm{i}0^+)x(1-x)}\right)
\end{aligned}\tag{37}$$

a result which is also valid for $i=1$. Let us warn the reader that we use the notation $\tilde p_i^{\mu}=\theta^{\mu\nu}p_{i\,\nu}$ and $\tilde p_i^2\equiv p_{i\,\mu}\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_{i\,\nu}$, so that $\tilde p_i^2\geqslant 0$. Substituting this result in equations (32), one comes to the conclusion that there is a pairwise cancellation mechanism at work:

$$\begin{aligned}
&p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\,j^{a_2\,+}_{\mu_2}(p_2)\,j^{a_3\,-}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}+p_3^{\mu_3}\,\langle j^{a_1\,+}_{\mu_1}(p_1)\,j^{a_2\,-}_{\mu_2}(p_2)\,j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}=0\\
&p_3^{\mu_3}\,\langle j^{a_1\,+}_{\mu_1}(p_1)\,j^{a_2\,-}_{\mu_2}(p_2)\,j^{a_3\,-}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}+p_3^{\mu_3}\,\langle j^{a_1\,-}_{\mu_1}(p_1)\,j^{a_2\,+}_{\mu_2}(p_2)\,j^{a_3\,+}_{\mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}=0.
\end{aligned}\tag{38}$$

Finally, taking into account equations (23), (30), (31) and (38), one concludes that in the pseudotensor part of $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\mathrm{NP}}$ no chiral gauge anomaly occurs, i.e.

$$p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)^{\mathrm{eps}}_{\mathrm{NP}}=0.$$

We have thus shown that a noncommutative $U(N)$ chiral theory with only chiral adjoint fermions does not present a chiral anomaly in the three-point function (triangle anomaly). The descent equations [5] lead to the conclusion that noncommutative $U(N)$ chiral gauge theory with only adjoint fermions is anomaly free.

Let us now study the gauge anomalies of the theory with action in equation (12). If this theory were gauge invariant at the quantum level the Ward identities should read thus:

$$\begin{aligned}
&\int\mathrm{d}^4x\;\omega^{i_1}_{\;i_2}\star\partial_\mu\frac{\delta\Gamma[A,B]}{\delta A^{i_1}_{\mu\,i_2}}=\mathrm{i}\int\mathrm{d}^4x\;\omega^{i_1}_{\;i_2}\star\left[A^{i_2}_{\mu\,i_3}\star\frac{\delta\Gamma[A,B]}{\delta A^{i_1}_{\mu\,i_3}}-\frac{\delta\Gamma[A,B]}{\delta A^{i_3}_{\mu\,i_2}}\star A^{i_3}_{\mu\,i_1}\right]\\
&\int\mathrm{d}^4x\;\chi^{j_1}_{\;j_2}\star\partial_\mu\frac{\delta\Gamma[A,B]}{\delta B^{j_1}_{\mu\,j_2}}=\mathrm{i}\int\mathrm{d}^4x\;\chi^{j_1}_{\;j_2}\star\left[B^{j_2}_{\mu\,j_3}\star\frac{\delta\Gamma[A,B]}{\delta B^{j_1}_{\mu\,j_3}}-\frac{\delta\Gamma[A,B]}{\delta B^{j_3}_{\mu\,j_2}}\star B^{j_3}_{\mu\,j_1}\right].
\end{aligned}\tag{39}$$

Let us introduce the following currents:

$$j^{a}_{\ \mu}(x) \equiv \mathrm{i}\frac{\delta S[A, B]}{A^a_\mu(x)} \qquad \text{and} \qquad j^{b}_{\ \mu}(x) \equiv \mathrm{i}\frac{\delta S[A, B]}{B^b_\mu(x)}.$$

Hence,

$$j^{a}_{\ \mu}(x) = -\mathrm{i}\, \psi^{j}_{\ k\,\beta} \star \bar{\psi}^{k}_{\ i\,\alpha}(x)\, T^{a\,i}_{U(N)\ j}\, \left(\bar{\gamma}^{\mu} P_{+}\right)_{\alpha\beta} \tag{40}$$

and

$$j^{b}_{\ \mu}(x) = -\mathrm{i}\, \bar{\psi}^{k}_{\ i\,\alpha} \star \psi^{i}_{\ j\,\beta}(x) T^{b\,j}_{U(M)\ k}\, \left(\bar{\gamma}^{\mu} P_{+}\right)_{\alpha\beta} \tag{41}$$

where $T^a_{U(N)}$ and $T^b_{U(M)}$ are the generators of $U(N)$ and $U(M)$ in the fundamental representation, respectively. We shall also need the nonsinglet currents

$$j^{(A)\,i_1}_{\mu\ \ i_2}(x) = -\mathrm{i}\, \psi^{i_1}_{\ \ j\,\beta} \star \bar{\psi}^{j}_{\ i_2\,\alpha}(x) \left(\bar{\gamma}^{\mu} P_{+}\right)_{\alpha\beta} \tag{42}$$

and

$$j^{(B)\,j_1}_{\mu\ \ j_2}(x) = -\mathrm{i}\, \bar{\psi}^{j_1}_{\ \ i\,\alpha} \star \psi^{i}_{\ j_2\,\beta}(x) \left(\bar{\gamma}^{\mu} P_{+}\right)_{\alpha\beta} \tag{43}$$

to express the rhs of equation (39) in terms of correlation functions of currents. Unlike the theories previously studied, now there are nonvanishing pseudotensor contributions to the two-point part of $\Gamma[A, B]$. These contributions enter the Ward identities in equation (39).

We have now the following independent three-current correlation functions:

$$\begin{gathered}
\langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}} \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}} \\
\langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}} \langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}} \\
\langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}} \text{ and } \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\mathrm{con}}.
\end{gathered} \tag{44}$$

The reader should bear in mind that the indices a_i, $i = 1$, 2 and 3, label currents of the type defined in equation (40), whereas if a current is of the type given in equation (41) it carries an index b_i, $i = 1$, 2 and 3. In equation (44) the first two correlation functions are sums of only planar triangle diagrams and the last four are sums of only nonplanar triangle diagrams. That there be no breaking of the classical gauge symmetry of the theory at hand in the triangle diagrams demands that the following equations hold:

$$\begin{aligned}
&p_3^{\mu_3}\, \langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0 \\
&p_3^{\mu_3}\, \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0 \\
&p_3^{\mu_3}\, \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0 \\
&p_3^{\mu_3}\, \langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0 \\
&p_3^{\mu_3}\, \langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} \\
&\qquad = -\,\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\, T^{a_1\,i_1}_{\ \ \ \ i_3}\, T^{a_3\,i_3}_{\ \ \ \ i_2}\, \langle j^{(A)\,i_2}_{\ \mu_1\,i_1}(-p_2)\, j^{b_2}_{\ \ \mu_2}(p_2)\,\rangle^{\mathrm{eps}}_{\mathrm{con}} \\
&\qquad\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)} T^{a_3\,i_1}_{\ \ \ \ i_3}\, T^{a_1\,i_3}_{\ \ \ \ i_2}\, \langle j^{(A)\,i_2}_{\ \mu_1\,i_1}(-p_2)\, j^{b_2}_{\ \ \mu_2}(p_2)\,\rangle^{\mathrm{eps}}_{\mathrm{con}} \\
&p_3^{\mu_3}\, \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} \\
&\qquad = -\,\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\, T^{b_1\,j_1}_{\ \ \ \ j_3}\, T^{b_3\,j_3}_{\ \ \ \ j_2}\, \langle j^{(B)\,j_2}_{\ \mu_1\,j_1}(-p_2)\, j^{a_2}_{\ \ \mu_2}(p_2)\,\rangle^{\mathrm{eps}}_{\mathrm{con}} \\
&\qquad\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)} T^{b_3\,j_1}_{\ \ \ \ j_3}\, T^{b_1\,j_3}_{\ \ \ \ j_2}\, \langle j^{(B)\,j_2}_{\ \mu_1\,j_1}(-p_2)\, j^{a_2}_{\ \ \mu_2}(p_2)\,\rangle^{\mathrm{eps}}_{\mathrm{con}}
\end{aligned} \tag{45}$$

where $p_3 = -p_1 - p_2$. The nonsinglet currents $j^{(A)}_\mu$ and $j^{(B)}_\mu$ are defined in equations (42) and (43), respectively. To obtain the previous equation, we have taken into account equation (39) and the result that the only two-point contribution to $\Gamma[A, B]$ which carries a pseudotensor contribution is of the type

$$\int \mathrm{d}^4x \int \mathrm{d}^4y\ \ \mathrm{Tr}\ A_{\mu_1}(x)\ \ \mathrm{Tr}\ B_{\mu_2}(y)\ f^{\mu_1\mu_2}\,(x, y|\theta)$$

with

$$f^{\mu_1\mu_2}(x, y|\theta) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} f^{\mu_1\mu_2}(p|\tilde{p})$$

$$f^{\mu_1\mu_2}(p|\tilde{p}) = \frac{i}{4\pi^2} \varepsilon_{\mu_1\mu_2\alpha\beta} \frac{\tilde{p}^\alpha p^\beta}{\tilde{p}^2} \times \int_0^1 dx \sqrt{\tilde{p}^2(-p^2 - i0^+)x(1-x)} \; K_1\left(\sqrt{\tilde{p}^2(-p^2 - i0^+)x(1-x)}\right).$$

This pseudotensor contribution is nonplanar and causes no anomaly.

Let us note that the first two identities in equation (45) do not hold, so they are anomalous, but that all the others do. The computations we have carried out for the theory with adjoint fermion fields can be readily adapted to the case at hand to obtain

$$\begin{aligned} p_3^{\mu_3}\, \langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} &= -(2\pi)^4 \delta(p_1+p_2+p_3) \frac{1}{24\pi^2} \varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta \\ &\quad \times M\big(\text{Tr}\,\{T^{a_1}_{U(N)}, T^{a_2}_{U(N)}\}\, T^{a_3}_{U(N)} \cos\tfrac{1}{2}\theta(p_1,p_2) \\ &\quad -i\,\text{Tr}\,[T^{a_1}_{U(N)}, T^{a_2}_{U(N)}]T^{a_3}_{U(N)} \sin\tfrac{1}{2}\theta(p_1,p_2)\big) \\ p_3^{\mu_3}\, \langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} &= (2\pi)^4 \delta(p_1+p_2+p_3) \frac{1}{24\pi^2} \varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta \\ &\quad \times N\big(\text{Tr}\,\{T^{b_1}_{U(M)}, T^{b_2}_{U(M)}\}\, T^{b_3}_{U(M)} \cos\tfrac{1}{2}\theta(p_1,p_2) \\ &\quad -i\,\text{Tr}\,[T^{b_1}_{U(M)}, T^{b_2}_{U(M)}]T^{b_3}_{U(M)} \sin\tfrac{1}{2}\theta(p_1,p_2)\big) \end{aligned} \tag{46}$$

and

$$\begin{aligned} &p_3^{\mu_3}\, \langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} \\ &\quad = -(2\pi)^4 \delta(p_1+p_2+p_3)\, \text{Tr}\,\big(T^{b_1}_{U(M)} T^{b_2}_{U(M)}\big)\, \text{Tr}\, T^{a_3}_{U(N)} \\ &\quad \times \big[e^{-\frac{i}{2}\theta(p_1,p_2)} \Delta^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3) + e^{\frac{i}{2}\theta(p_1,p_2)} \Delta^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)\big] \\ &p_3^{\mu_3}\, \langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} \\ &\quad = (2\pi)^4 \delta(p_1+p_2+p_3)\, \text{Tr}\,\big(T^{a_1}_{U(N)} T^{a_2}_{U(N)}\big)\, \text{Tr}\, T^{b_3}_{U(M)} \\ &\quad \times \big[e^{\frac{i}{2}\theta(p_1,p_2)} \Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3) + e^{-\frac{i}{2}\theta(p_1,p_2)} \Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)\big] \\ &p_3^{\mu_3}\, \langle j^{a_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} \\ &\quad = (2\pi)^4 \delta(p_1+p_2+p_3)\, \text{Tr}\,\big(T^{a_1}_{U(N)} T^{a_3}_{U(N)}\big)\, \text{Tr}\, T^{b_2}_{U(M)} \\ &\quad \times \big[e^{-\frac{i}{2}\theta(p_1,p_2)} \Delta^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2) + e^{\frac{i}{2}\theta(p_1,p_2)} \Delta^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)\big] \\ &p_3^{\mu_3}\, \langle j^{b_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\text{eps}}_{\text{con}} \\ &\quad = -(2\pi)^4 \delta(p_1+p_2+p_3)\, \text{Tr}\,\big(T^{b_1}_{U(M)} T^{b_3}_{U(M)}\big)\, \text{Tr}\, T^{a_2}_{U(N)} \\ &\quad \times \big[e^{\frac{i}{2}\theta(p_1,p_2)} \Delta^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2) + e^{-\frac{i}{2}\theta(p_1,p_2)} \Delta^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)\big]. \end{aligned} \tag{47}$$

From equation (46), one deduces that the anomaly cancellation condition for the planar triangle diagrams reads

$$\text{Tr}\,\big(T^{a_1}_{U(N)} T^{a_2}_{U(N)} T^{a_3}_{U(N)}\big) = 0 \qquad \text{and} \qquad \text{Tr}\,\big(T^{b_1}_{U(M)} T^{b_2}_{U(M)} T^{b_3}_{U(M)}\big) = 0.$$

Both the anomalies which give rise to these anomaly cancellation conditions are analogous to the anomaly in equation (10), i.e. the anomaly for chiral fundamental fermions. If we now substitute equation (37) in (47), we shall conclude that the left-hand sides of the last two identities in equation (45) do not vanish, but read, respectively, thus:

$$
\begin{aligned}
& p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} \\
&\quad = (2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}\,\big(T^{a_1}_{U(N)}T^{a_3}_{U(N)}\big)\,\mathrm{Tr}\;T^{b_2}_{U(M)} \\
&\qquad \times\Bigg[\frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\;\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde p_2^{\alpha}p_2^{\beta}}{\tilde p_2^2} \\
&\qquad \times\int_0^1 \mathrm{d}x\,\sqrt{\tilde p_2^2(-p_2^2-\mathrm{i}0^+)x(1-x)}\;\;K_1\bigg(\sqrt{\tilde p_2^2(-p_2^2-\mathrm{i}0^+)x(1-x)}\bigg)\Bigg] \\
& p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} \\
&\quad = -\,(2\pi)^4\delta(p_1+p_2+p_3)\,\mathrm{Tr}\,\big(T^{b_1}_{U(M)}T^{b_3}_{U(M)}\big)\,\mathrm{Tr}\;T^{a_2}_{U(N)} \\
&\qquad \times\Bigg[\frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\;\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde p_2^{\alpha}p_2^{\beta}}{\tilde p_2^2} \\
&\qquad \times\int_0^1 \mathrm{d}x\,\sqrt{\tilde p_2^2(-p_2^2-\mathrm{i}0^+)x(1-x)}\;\;\mathrm{K}_1\bigg(\sqrt{\tilde p_2^2(-p_2^2-\mathrm{i}0^+)x(1-x)}\bigg)\Bigg].
\end{aligned}
\tag{48}
$$

Recall that $\tilde p_2^{\mu}=\theta^{\mu\nu}p_{2\,\nu}$ and $\tilde p_2^2\equiv p_{2\,\mu}\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_{2\,\nu}$, so $\tilde p_2^2\geqslant 0$.

Finally, equation (26) implies that

$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con}=0.$$

Similarly,

$$p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con}=0.$$

To show that indeed the last four identities in equation (45) hold, all that remains for us to do is to work out the following expressions:

$$
\begin{aligned}
&-\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\,T^{a_1}{}^{i_1}_{i_3}\,T^{a_3}{}^{i_3}_{i_2}\,\langle j^{(A)}{}^{i_2}_{\mu_1\,i_1}(-p_2)\, j^{b_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con} \\
&\qquad +\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}T^{a_3}{}^{j_1}_{j_3}\,T^{a_1}{}^{j_3}_{j_2}\,\langle j^{(A)}{}^{j_2}_{\mu_1\,j_1}(-p_2)\, j^{b_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con} \\
&-\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(p_1,p_2)}\,T^{b_1}{}^{j_1}_{j_3}\,T^{b_3}{}^{j_3}_{j_2}\,\langle j^{(B)}{}^{j_2}_{\mu_1\,j_1}(-p_2)\, j^{a_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con} \\
&\qquad +\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta(p_1,p_2)}T^{b_3}{}^{j_1}_{j_3}\,T^{b_1}{}^{j_3}_{j_2}\,\langle j^{(B)}{}^{j_2}_{\mu_1\,j_1}(-p_2)\, j^{a_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}.
\end{aligned}
$$

It is not difficult to see that the previous expressions are equal to

$$
\begin{aligned}
&-\mathrm{i}\;\mathrm{Tr}\,\big(T^{a_1}_{U(N)}T^{a_3}_{U(N)}\big)\,\mathrm{Tr}\;T^{b_2}_{U(M)}\;\sin\frac{1}{2}\theta(p_1,p_2)\int\frac{\mathrm{d}q^4}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\theta(q,p_2)}\,\frac{\mathrm{tr}\{(\not q+\not p_2)\gamma_{\mu_2}\not q\gamma_{\mu_1}\gamma_5\}}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)} \\
&-\mathrm{i}\;\mathrm{Tr}\,\big(T^{b_1}_{U(M)}T^{b_3}_{U(M)}\big)\,\mathrm{Tr}\;T^{a_2}_{U(N)}\;\sin\frac{1}{2}\theta(p_1,p_2)\int\frac{\mathrm{d}q^4}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\theta(q,p_2)}\,\frac{\mathrm{tr}\{(\not q+\not p_2)\gamma_{\mu_2}\not q\gamma_{\mu_1}\gamma_5\}}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)}
\end{aligned}
$$

respectively. Some algebra and the help of the appendix makes it possible for us to conclude that the right-hand sides of the last two identities in equation (45) agree, respectively, with their left-hand sides, the latter being given in equation (48).

In summary, we have shown that the last four identities of equation (45) indeed hold in the quantum theory. These identities are the Ward identities for the nonplanar contributions to the three-point function of $\Gamma[A,B]$: the Ward identities for the nonplanar triangle contributions. Hence, the nonplanar triangle contributions give rise to no gauge anomaly. On the other hand, the planar triangle contributions are anomalous with the anomalies given in equation (46).

3. The IR origin of non-Abelian chiral gauge anomalies

In the previous section we have shown that, for the theories we are discussing, only planar triangle diagrams give rise to a gauge anomaly and we have given a UV interpretation of this anomaly. Equation (10) is the basic building-block for this type of anomaly: see equation (46). To interpret the non-Abelian chiral anomaly under scrutiny as an IR phenomenon, we shall follow Coleman and Grossman [7] and compute $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ at the point

$$p_1^2=p_2^2=p_3^2=-Q^2 \qquad p_1+p_2+p_3=0.$$

$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ is the pseudotensor part of the three-point function for a noncommutative gauge theory with a right-handed fundamental fermion. The action of this theory is given in equation (5). The corresponding IR analysis for the planar triangle diagrams arising in the other theories studied in this paper (see equations (6), (7) and (12)) can be readily performed by adapting the results presented in the following.

Let us recall first that formally $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ is given by the sum of the pseudotensor contributions from the triangle diagrams in figure 2, which for the case at hand reads

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps} = {}& {\rm e}^{-\frac{\rm i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_1}T^{a_2}T^{a_3}\ \Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)\\ &+{\rm e}^{\frac{\rm i}{2}\theta(p_1,p_2)}\ {\rm Tr}\ T^{a_2}T^{a_1}T^{a_3}\ \Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)\end{aligned}\tag{49}$$

where

$$\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2) = \int \frac{{\rm d}^4q}{(2\pi)^4}\,\frac{{\rm tr}^{\rm eps}\left\{(\not q+\not p_1)\,\gamma_{\mu_1}P_+\not q\gamma_{\mu_2}P_+\,(\not q-\not p_2)\,\gamma_{\mu_3}P_+\right\}}{(q^2+{\rm i}0^+)((q+p_1)^2+{\rm i}0^+)((q-p_2)^2+{\rm i}0^+)}$$

and

$$\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2) = \int \frac{{\rm d}^4q}{(2\pi)^4}\,\frac{{\rm tr}^{\rm eps}\left\{(\not q+\not p_2)\,\gamma_{\mu_2}P_+\not q\gamma_{\mu_1}P_+\,(\not q-\not p_1)\,\gamma_{\mu_3}P_+\right\}}{(q^2+{\rm i}0^+)((q+p_2)^2+{\rm i}0^+)((q-p_1)^2+{\rm i}0^+)}.$$

The symbol ${\rm tr}^{\rm eps}$ denotes the pseudotensor contributions, i.e. contributions involving an odd number of γ_5 matrices.

As they stand the Feynman amplitudes $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ and $\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ above are at first sight formal expressions since they are sums of Feynman integrals UV divergent by power-counting. However, we shall see in a moment that one can associate with these Feynman amplitudes a unique tempered distribution provided cyclicity of the external indices and momenta is imposed. Indeed, renormalization theory [14] associates with every formal Feynman amplitude a tempered distribution which is uniquely defined up to a local polynomial of the appropriate dimension in the external momenta[1]. This polynomial can be further restricted by symmetries. Hence, the Feynman amplitude $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ can be uniquely defined as a distribution modulo the following polynomial:

$$C_1\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,p_1^\alpha\ +\ C_2\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,p_2^\alpha \tag{50}$$

where C_1 and C_2 are arbitrary constants. If we next impose symmetry under cyclic permutations of the pairs (μ_1,p_1), (μ_2,p_2), (μ_3,p_3), with $p_1+p_2+p_3=0$, then C_1 and C_2 are fixed once and for all. Indeed, any further addition ought to be of the type

$$C_3\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,(p_1\ +\ p_2\ +\ p_3)^\alpha$$

which vanishes upon imposing four-momentum conservation. Actually, what this discussion is telling us is that if we use, as an intermediate computational procedure, a regularization method that explicitly preserves the formal symmetry of $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ under cyclic permutations of the pairs (μ_1,p_1), (μ_2,p_2), (μ_3,p_3), the limit in which the regulator is removed is well defined. Besides, this limit is the same for all regularizations (and, of course, renormalizations) of $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ which preserve its formal cyclic symmetry. Of course, any renormalization

[1] Here we assume that, since the diagrams we are considering are one loop and planar, standard renormalization theory can be applied to each diagram without further ado.

which breaks this cyclic symmetry can be brought to the unique symmetric form just mentioned by adding a finite counterterm of the form given in equation (50). It is in this sense that we are entitled to say that the Feynman amplitude $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1, p_2)$ is a UV finite quantity, in spite of the fact that it is not UV finite by power-counting. The same kind of argument can be applied to $\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1, p_2)$ to conclude that it is also a UV finite object, though it is not UV finite by power-counting.

There is a very handy regularization procedure which explicitly preserves the symmetry of each triangle diagram in figure 2 under cyclic permutations of its external legs. This is the dimensional regularization algorithm set up in the previous section. The dimensionally regularized counterparts of $\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1, p_2)$ and $\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1, p_2)$ read

$$\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1, p_2; d) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}^{\mathrm{eps}}\left\{(\not q + \not p_1)\, \bar\gamma_{\mu_1} P_+ \not q \bar\gamma_{\mu_2} P_+ (\not q - \not p_2)\, \bar\gamma_{\mu_3} P_+\right\}}{(q^2+\mathrm{i}0^+)((q+p_1)^2+\mathrm{i}0^+)((q-p_2)^2+\mathrm{i}0^+)}$$

and

$$\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1, p_2; d) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}^{\mathrm{eps}}\left\{(\not q + \not p_2)\, \bar\gamma_{\mu_2} P_+ \not q \bar\gamma_{\mu_1} P_+ (\not q - \not p_1)\, \bar\gamma_{\mu_3} P_+\right\}}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)((q-p_1)^2+\mathrm{i}0^+)}. \qquad (51)$$

Taking into account that

$$P_+\hat\gamma_\mu\bar\gamma_\nu P_+ = 0 \qquad P_+\bar\gamma_\mu\bar\gamma_\nu = \bar\gamma_\mu\bar\gamma_\nu P_+$$

we conclude that equation (51) can be turned into the following one:

$$\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1, p_2; d) = \frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}\left\{(\bar{\not q} + \bar{\not p}_1)\, \bar\gamma_{\mu_1} \bar{\not q} \bar\gamma_{\mu_2} (\bar{\not q} - \bar{\not p}_2)\, \bar\gamma_{\mu_3}\gamma_5\right\}}{(q^2+\mathrm{i}0^+)((q+p_1)^2+\mathrm{i}0^+)((q-p_2)^2+\mathrm{i}0^+)}$$

and

$$\Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1, p_2; d) = \frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{tr}\left\{(\bar{\not q} + \bar{\not p}_2)\, \bar\gamma_{\mu_2} \bar{\not q} \bar\gamma_{\mu_1} (\bar{\not q} - \bar{\not p}_1)\, \bar\gamma_{\mu_3}\gamma_5\right\}}{(q^2+\mathrm{i}0^+)((q+p_2)^2+\mathrm{i}0^+)((q-p_1)^2+\mathrm{i}0^+)}. \qquad (52)$$

The computation of the previous integrals at $p_1^2 = p_2^2 = p_3^2 = -Q^2$ is very easy. The substitution in equation (52) of the integrals in the appendix and some self-evident algebraic arrangements yield upon taking the limit $d \to 4$ the following result:

$$\begin{aligned}\Delta^{(1)}_{\mu_1\mu_2\mu_3}(p_1, p_2) &= \Delta^{(2)}_{\mu_1\mu_2\mu_3}(p_1, p_2)\\ &= \frac{1}{24\pi^2}\left(\frac{1}{Q^2}\right)\left(\varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta\, p_{3\,\mu_3} + \varepsilon_{\mu_3\mu_1\alpha\beta}\, p_3^\alpha p_1^\beta\, p_{2\,\mu_2} + \varepsilon_{\mu_2\mu_3\alpha\beta}\, p_2^\alpha p_3^\beta\, p_{1\,\mu_1}\right).\end{aligned} \qquad (53)$$

The Feynman amplitudes in the previous equation have poles at $Q^2 = 0$ and it is these IR singularities which we shall hold responsible for the existence of the non-Abelian chiral anomaly [7]. If we now substitute equations (53) into (49) we shall obtain the whole anomalous contribution to the three-point function at $p_1^2 = p_2^2 = p_3^2 = -Q^2$:

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}&(p_1, p_2)^{\mathrm{eps}}\\ &= \frac{1}{24\pi^2}\left(\frac{1}{Q^2}\right)\left(\mathrm{Tr}\,\{T^{a_1}, T^{a_2}\}\, T^{a_3}\cos\tfrac{1}{2}\theta(p_1, p_2)\right.\\ &\quad \left. -\mathrm{i}\,\mathrm{Tr}\,[T^{a_1}, T^{a_2}]T^{a_3}\sin\tfrac{1}{2}\theta(p_1, p_2)\right)\\ &\quad \times\left(\varepsilon_{\mu_1\mu_2\alpha\beta}\, p_1^\alpha p_2^\beta\, p_{3\,\mu_3} + \varepsilon_{\mu_3\mu_1\alpha\beta}\, p_3^\alpha p_1^\beta\, p_{2\,\mu_2} + \varepsilon_{\mu_2\mu_3\alpha\beta}\, p_2^\alpha p_3^\beta\, p_{1\,\mu_1}\right).\end{aligned} \qquad (54)$$

Notice that by contracting with $p_3^{\mu_3}$ both sides of the previous equation, one obtains once again the anomaly equation (equation (10)). Also notice that unlike in the commutative case the rhs of equation (54) vanishes if and only if $\mathrm{Tr}\,\{T^{a_1}, T^{a_2}\}\, T^{a_3} = 0$ and $\mathrm{Tr}\,[T^{a_1}, T^{a_2}]T^{a_3} = 0$, i.e. $\mathrm{Tr}\;\; T^{a_1}T^{a_2}T^{a_3} = 0$. Indeed, the nonpolynomial—in the Moyal product—IR contributions,

$$\frac{\cos\frac{1}{2}\theta(p_1,p_2)}{Q^2} \qquad \text{and} \qquad \frac{\sin\frac{1}{2}\theta(p_1,p_2)}{Q^2}$$

in this equation make it impossible for us to redefine $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\mathrm{eps}}$, so the anomaly cancellation condition reads merely $\mathrm{Tr}\,\{T^{a_1}, T^{a_2}\}\, T^{a_3} = 0$.

4. Summary and conclusions

In this paper we have shown that the one-loop noncommutative non-Abelian gauge anomalies for $U(N)$ groups can be interpreted either as a UV effect or as an IR phenomenon. We have considered three basic types of noncommutative chiral gauge theory, namely, gauge theories with a fundamental, gauge theories with an adjoint and gauge theories with a bi-fundamental right-handed fermion. We have computed the anomaly in one-loop planar triangle diagrams and shown that the nonplanar contributions yield no gauge anomaly since they preserve the corresponding Ward identities. It turned out that chiral gauge theories with fundamental, anti-fundamental and bi-fundamental matter are, in general, anomalous and that chiral theories with only adjoint fermions are always anomaly free. Last but not least, we have clarified the origin of the noncommutative anomaly cancellation condition $\mathrm{Tr}\;\; T^{a_1}T^{a_2}T^{a_3} = 0$.

It will be interesting to carry out the analysis presented here for the theories introduced in [15] and for the axial anomaly [16]. Anomalies in the presence of noncommutative gravity [17] are also worth studying. We shall report on these topics elsewhere.

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Appendix

The following result is needed to obtain equation (37):

$$\begin{aligned}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \mathrm{e}^{\pm \mathrm{i}\theta(q,p)}\, &\frac{q^\mu}{(q^2+\mathrm{i}0^+)((q+p)^2+\mathrm{i}0^+)} \\ &= -\frac{\mathrm{i}\,p^\mu}{8\pi^2}\int_0^1 \mathrm{d}x\; x\; K_0\left(\sqrt{\tilde p^2(-p^2-\mathrm{i}0^+)x(1-x)}\right) \\ &\quad \pm \frac{1}{8\pi^2}\frac{\tilde p^\mu}{\tilde p^2}\int_0^1 \mathrm{d}x\, \sqrt{\tilde p^2(-p^2-\mathrm{i}0^+)x(1-x)} \\ &\quad \times K_1\left(\sqrt{\tilde p^2(-p^2-\mathrm{i}0^+)x(1-x)}\right)\end{aligned}$$

where $\tilde p^\mu = \theta^{\mu\nu}p_\nu$, but $\tilde p^2 \equiv p_\mu\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_\nu$, so that $\tilde p_i^2 \geqslant 0$.

Next, we display the integrals needed to obtain equation (53). These integrals are worked out at the point $p_1^2 = p_2^2 = -2\,p_1\cdot p_2 = -\,Q^2$. They read

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 (q+p_1)^2 (q-p_2)^2} = \frac{\Phi}{Q^2} + O(d-4)$$
$$\int \frac{d^d q}{(2\pi)^d} \frac{\bar{q}_\mu}{q^2 (q+p_1)^2 (q-p_2)^2} = \left(\frac{\Phi}{3}\right)\left(\frac{1}{Q^2}\right)(\bar{p}_2 - \bar{p}_1)_\mu + O(d-4)$$
$$\int \frac{d^d q}{(2\pi)^d} \frac{\bar{q}_\mu \bar{q}_\nu}{q^2 (q+p_1)^2 (q-p_2)^2} = \left(\frac{I_1}{4} + \frac{\Phi}{6} + \frac{I_2}{4}\right)\bar{g}_{\mu\nu} + \frac{I_2}{6}\left(\frac{1}{Q^2}\right)\left(\bar{p}_{1\,\mu}\bar{p}_{2\,\nu} + \bar{p}_{2\,\mu}\bar{p}_{1\,\nu}\right)$$
$$+\left(\frac{\Phi}{3} + \frac{I_2}{3}\right)\left(\frac{1}{Q^2}\right)\left(\bar{p}_{1\,\mu}\bar{p}_{1\,\nu} + \bar{p}_{2\,\mu}\bar{p}_{2\,\nu}\right) + O(d-4)$$
$$\int \frac{d^d q}{(2\pi)^d} \frac{\bar{q}^2\,\bar{q}_\mu}{q^2 (q+p_1)^2 (q-p_2)^2} = \left(\frac{I_1}{2} + \frac{I_2}{6}\right)(\bar{p}_2 - \bar{p}_1)_\mu + O(d-4)$$

where

$$I_1 = \frac{i}{16\pi^2}\left(-\frac{1}{\epsilon} - \gamma - \ln\frac{Q^2}{4\pi\kappa^2} + 2\right)$$
$$I_2 = \frac{i}{16\pi^2}$$
$$\Phi = \frac{i}{16\pi^2}\int_0^1 dx\,\frac{\ln x(1-x)}{1-x+x^2}$$

and $d = 4 + 2\epsilon$.

Note that all the ugly features of the integrals above nicely cancel against one another when substituted in equation (52) to yield the beautiful result of equation (53).

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